

## Surfaces, circles, and solenoids

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August 23, 2004

**Abstract** Previous work of the author has developed coordinates on bundles over the classical Teichmüller spaces of punctured surfaces and on the space of cosets of the Möbius group in the group of orientation-preserving homeomorphisms of the circle, and this work is surveyed here. Joint work with Dragomir Šarić is also sketched which extends these results to the setting of the Teichmüller space of the solenoid of a punctured surface, which is defined here in analogy to Dennis Sullivan’s original definition for the case of closed surfaces. Because of relations with the classical modular group, the punctured solenoid and its Teichmüller theory have connections with number theory.

## Introduction

The “lambda length” of a pair of horocycles in upper halfspace centered at  $u, v \in \mathbf{R}$  of respective diameters  $c, d$  is given by  $\sqrt{\frac{2}{cd}} |u - v|$  and is roughly the hyperbolic distance between them. (See §1 for more precision.) These invariants can be used to devise coordinates in several different guises: for the Teichmüller space of punctured surfaces [7]; for the space of cosets of the Möbius group of real fractional linear transformations in the topological group of all orientation-preserving homeomorphisms of the circle, which forms a generalized universal Teichmüller space [8]; and for the Teichmüller space of the “punctured solenoid”, which is the punctured analogue introduced in [10] (and defined in §4) of the space studied by Sullivan [12] to analyze dynamical properties of the mapping class group actions on the Teichmüller spaces for closed surfaces. In fact in each case, lambda lengths give coordinates for the “decorated” Teichmüller space rather than the Teichmüller space. (The respective notions of decoration are defined in §§ 2,3,4.) Furthermore, the manifestation of lambda lengths as coordinates on the decorated Teichmüller space of the

punctured solenoid is the first step of a larger ongoing program with Šarić [10] to extend the decorated Teichmüller theory [7]–[9] to the solenoid.

To define the punctured solenoid  $\mathcal{H}$  as a topological space, for definiteness fix the “modular” group  $G = PSL_2(\mathbf{Z})$  of integral fractional linear transformations, let  $\hat{G}$  denote its pro-finite completion (whose definition is recalled in §4), let  $\mathbf{D}$  denote the open unit disk in the complex plane, and define  $\mathcal{H} = (\mathbf{D} \times \hat{G})/G$ , where  $\gamma \in G$  acts on  $(z, t) \in \mathbf{D} \times \hat{G}$  by  $\gamma(z, t) = (\gamma z, t\gamma^{-1})$ . In analogy to the case of punctured surfaces, we may produce appropriate geometric structures on  $\mathcal{H}$  by taking suitable quotients  $(\mathbf{D} \times \hat{G})/G$  by other actions of  $G$  on  $\mathbf{D} \times \hat{G}$ . As a pro-finite completion, the punctured solenoid itself is defined essentially number theoretically in terms of finite-index subgroups of the modular group, and aspects of its Teichmüller theory bear close relation to classical questions in number theory (as mentioned at the end of §5, which also contains other concluding and speculative remarks).

We take this opportunity to correct Theorem 6.4 from [8]. See the remarks following Theorem 8 for the correction to the universal Teichmüller theory and Proposition 12 for the corresponding affirmative statement for the solenoid.

**Acknowledgement** It was discussions with Mahmoud Zeinalian that led to the original idea of employing lambda lengths for solenoids, and is a pleasure to thank him, as well as Bob Guralnick for useful comments on classical number theory. The new material in §4 on the punctured solenoid is joint work [10] with Dragomir Šarić, who patiently explained his earlier work [11], and it is also a pleasure to thank him for many stimulating conversations.

## 1. Background

Define the Minkowski inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbf{R}^3$  whose quadratic form is given by  $x^2 + y^2 - z^2$  in the usual coordinates. The upper sheet

$$\mathbf{H} = \{u = (x, y, z) \in \mathbf{R}^3 : \langle u, u \rangle = -1 \text{ and } z > 0\}$$

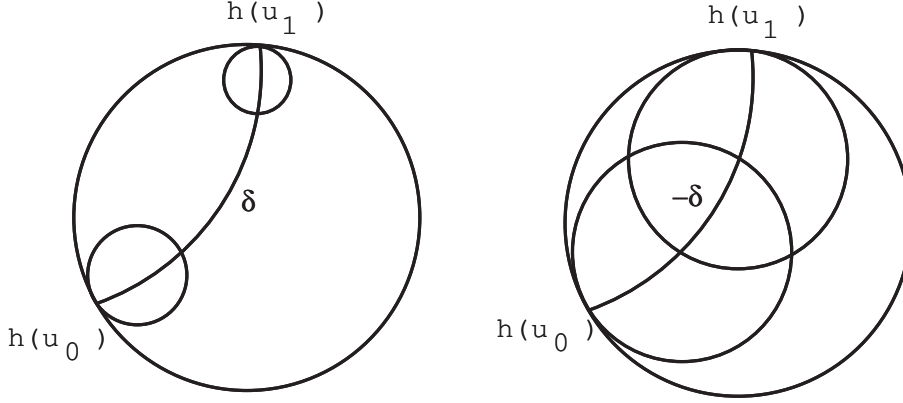
of the two-sheeted hyperboloid is isometric to the hyperbolic plane. Indeed, identifying the Poincaré disk  $\mathbf{D}$  with the open unit disk at height zero about the origin in  $\mathbf{R}^3$ , central projection  $\mathbf{H} \rightarrow \mathbf{D}$  from  $(0, 0, -1) \in \mathbf{R}^3$  establishes an isometry. Moreover, the open positive light cone

$$L^+ = \{u = (x, y, z) \in \mathbf{R}^3 : \langle u, u \rangle = 0 \text{ and } z > 0\}$$

is identified with the collection of all horocycles in  $\mathbf{H}$  via the affine duality  $u \mapsto h(u) = \{w \in \mathbf{H} : \langle w, u \rangle = -1\}$ . Identifying the unit circle  $S^1$  with the boundary of  $\mathbf{D}$ , the central projection extends continuously to the projection  $\Pi : L^+ \rightarrow S^1$  which maps a horocycle in  $L^+$  to its center in  $S^1$ .

Define a “decorated geodesic” to be an unordered pair  $\{h_0, h_1\}$  of horocycles with distinct centers in the hyperbolic plane, so there is a well-defined geodesic connecting the

centers of  $h_0$  and  $h_1$ ; the two horocycles may or may not be disjoint, and there is a well-defined signed hyperbolic distance  $\delta$  between them (taken to be positive if and only if  $h_0 \cap h_1 = \emptyset$ ) as illustrated in the two cases of Figure 1. The *lambda length* of the decorated geodesic  $\{h_0, h_1\}$  is defined to be the transform  $\lambda(h_0, h_1) = \sqrt{2 \exp \delta}$ . Taking this particular transform renders the identification  $h$  geometrically natural in the sense that  $\lambda(h(u_0), h(u_1)) = \sqrt{-\langle u_0, u_1 \rangle}$ , for  $u_0, u_1 \in L^+$  as one can check.



**Figure 1** Decorated geodesics.

Three useful lemmas (with computational proofs, which we do not reproduce here) are as follows:

**Lemma 1** [7;Lemma 2.4] *Given three rays  $\vec{r}_0, \vec{r}_1, \vec{r}_2 \subseteq L^+$  from the origin which contain linearly independent vectors and given three numbers  $\lambda_0, \lambda_1, \lambda_2 \in \mathbf{R}_+$ , there are unique points  $u_i \in \vec{r}_i$ , for  $i = 0, 1, 2$ , so that  $\lambda(h(u_i), h(u_j)) = \lambda_k$ , where  $\{i, j, k\} = \{0, 1, 2\}$ . The points  $u_0, u_1, u_2$  depend continuously on  $\lambda_0, \lambda_1, \lambda_2$  and on  $\vec{r}_0, \vec{r}_1, \vec{r}_2$ .*

**Lemma 2** [7;Lemma 2.3] *Given two points  $u_0, u_1 \in L^+$ , which do not lie on a common ray through the origin, and given two numbers  $\lambda_0, \lambda_1 \in \mathbf{R}_+$ , there is a unique point  $v \in L^+$  on either side of the plane through the origin containing  $u_0, u_1$  satisfying  $\lambda(h(v), h(u_i)) = \lambda_i$ , for  $i = 0, 1$ . The point  $v$  depends continuously on  $u_0, u_1$  and on  $\lambda_0, \lambda_1$ .*

**Lemma 3** [7;Proposition 2.8] *Suppose that  $u_0, u_1, u_2 \in L^+$  are linearly independent, let  $\gamma(u_i, u_j)$  denote the geodesic in  $\mathbf{H}$  with ideal vertices given by the centers of  $h(u_i)$  and  $h(u_j)$ , for  $i \neq j$ , and define*

$$-\lambda_i^2 = \langle u_j, u_k \rangle, \quad \alpha_i = \frac{\lambda_i}{\lambda_j \lambda_k}, \quad \text{for } \{i, j, k\} = \{0, 1, 2\}.$$

*Then  $2\alpha_i$  is the hyperbolic length along the horocycle  $h(u_i)$  between  $\gamma(u_i, u_j)$  and  $\gamma(u_i, u_k)$ , for  $\{i, j, k\} = \{0, 1, 2\}$ .*

**Remark 1** Consider an ideal quadrilateral  $Q$  in  $\mathbf{D}$  decorated so as to give four points in  $L^+$ . The edges of  $Q$  have well defined lambda lengths, say  $a, b, c, d$  in correct cyclic

(clockwise) order about the boundary of  $Q$ . Choose a diagonal of  $Q$ , where the diagonal has lambda length  $e$  and separates edges with lambda lengths  $a, b$  from edges with lambda lengths  $c, d$ . The other diagonal of  $Q$  has its lambda length  $f$  given by

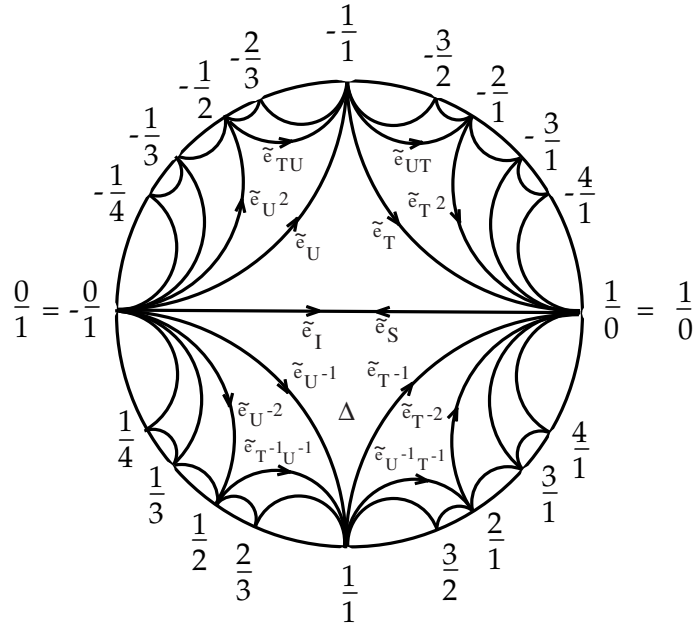
$$ef = ac + bd,$$

and we say that  $f$  arises from  $e$  by a *Ptolemy transformation* on the lambda lengths. To see this, note that the formula for a Ptolemy transformation is independent under scaling any of the four points in  $L^+$ , so we may alter the decoration and assume that the four points lie in a common horizontal plane. In this plane, the Minkowski inner product induces a multiple of the usual Euclidean metric, and the intersection of  $L^+$  with this plane is a round circle. The formula for the Ptolemy transformation thus follows from Ptolemy's classical formula on Euclidean lengths of quadrilaterals that inscribe in a circle.

**Remark 2** Consider a decorated triangle, say with lambda lengths  $x, y, z$  in the cyclic order about the boundary of the triangle determined by an orientation, and define a 2-form

$$\omega(x, y, z) = d\ln x \wedge d\ln y + d\ln y \wedge d\ln z + d\ln z \wedge d\ln x.$$

A calculation shows that  $\omega(a, b, e) + \omega(c, d, e) = \omega(b, c, f) + \omega(d, a, f)$ , thus assigning a well-defined Ptolemy-invariant 2-form to an oriented decorated ideal quadrilateral.



**Figure 2**

Regard the Poincaré disk as the open unit disk  $\mathbf{D}$  in the complex plane in the usual way so that the unit circle  $S^1$  is identified with the circle at infinity, and let  $\Delta$  denote the ideal hyperbolic triangle with vertices  $+1, -1, -\sqrt{-1} \in S^1$  as in Figure 2. Let  $\Gamma$  denote the group generated by reflections in the sides of  $\Delta$ , and define the *Farey tessellation*  $\tau_*$  to be the full  $\Gamma$ -orbit of the frontier of  $\Delta$ . We refer to geodesics in  $\tau_*$  as *edges* of  $\tau_*$  and think of

$\tau_*$  itself as a set of edges. The ideal vertices of the edges of  $\tau_*$  are naturally identified with the set  $\mathbf{Q}$  of all rational numbers including infinity, where for instance  $+1, -1, -\sqrt{-1} \in S^1$  correspond respectively to  $\infty = \frac{1}{0}, 0 = \frac{0}{1}, 1 = \frac{1}{1}$  as illustrated in Figure 2. Let  $\mathbf{Q} \subseteq S^1$  denote the corresponding countable dense subset of  $S^1$  which we refer to simply as the set of *rational points* of  $S^1$ . Define the *distinguished oriented edge* or *doe* of the Farey tessellation to be the oriented edge from  $\frac{0}{1}$  to  $\frac{1}{0}$ .

The *modular group*  $PSL_2 = PSL_2(\mathbf{Z})$  of integral fractional linear transformations is the subgroup of  $\Gamma$  consisting of compositions of an even number of reflections, and  $PSL_2$  acts simply transitively on the set of orientations on the edges of  $\tau_*$ . The assignment

$$e_A = A(doe), \text{ for } A \in PSL_2,$$

establishes a bijection between  $PSL_2$  and the set of oriented edges of  $\tau_*$  as illustrated in Figure 2. In particular, the doe of  $\tau_*$  is  $e_I$ , where  $I$  denotes the identity of  $PSL_2$ .

We adopt the standard notation

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

for certain elements of  $PSL_2$ , where  $S$  is involutive and fixes the unoriented edge of  $\tau_*$  underlying the doe while changing its orientation, and  $U$  (respectively  $T$ ) is the parabolic transformation with fixed point  $\frac{0}{1}$  (respectively  $\frac{1}{0}$ ) which cyclically permutes the incident edges of  $\tau_*$  in the counter-clockwise sense about  $\frac{0}{1}$  (respectively the clockwise sense about  $\frac{1}{0}$ ). In fact,  $U^{-1} = STS$ ,  $T^{-1} = SUS$ , and any two of  $S, T, U$  generate  $PSL_2$ .

We shall also require the full *Möbius group*  $Möb = PSL_2(\mathbf{R}) \supseteq PSL_2(\mathbf{Z}) = PSL_2$  consisting of all real fractional linear transformations.

## 2. Punctured surfaces

Let  $F = F_g^s$  denote a fixed smooth surface of genus  $g$  with  $s \geq 1$  punctures, where  $2 - 2g - s < 0$ .

Choose any base-point to determine the fundamental group  $G$  of  $F$ , and consider the space  $Hom'(G, Möb)$  of all discrete and faithful representations  $\rho : G \rightarrow Möb$  so that no holonomy  $\rho(\gamma)$  is elliptic for  $\gamma \in G$ , and the holonomies around the punctures of  $F$  are parabolic. Define the *Teichmüller space*

$$T(F) = Hom'(G, Möb)/Möb,$$

where  $Möb$  acts by conjugacy.

If  $\rho \in Hom'(G, Möb)$ , then  $\mathbf{D}/\rho(G)$  induces a complete finite-area hyperbolic structure on  $F$ , whose punctures are in one-to-one correspondence with the  $G$ -orbits of the set of fixed points of parabolic elements of  $\rho(G)$ .

Define the *decorated Teichmüller space*  $\tilde{T}(F) \rightarrow T(F)$  of  $F$  to be the trivial  $R_{>0}^s$ -bundle, where the fiber over a point is the set of all  $s$ -tuples of horocycles, one horocycle about each puncture of  $F$  parametrized by hyperbolic length

By an *arc family* in  $F$ , we mean the isotopy class of a family of essential arcs disjointly embedded in  $F$  connecting punctures, where no two arcs in a family may be homotopic rel punctures. If  $\alpha$  is a maximal arc family, so that each component of  $F - \cup \alpha$  is a triangle, then we say that  $\alpha$  is an *ideal triangulation* of  $F$ .

**Theorem 4** [7; Theorem 3.1] *Fix an ideal triangulation  $\tau$  of  $F$ . Then the assignment of  $\lambda$ -lengths  $\tilde{T}(F) \rightarrow R_{>0}^\tau$  is a homeomorphism onto.*

**Proof** We must describe an inverse to the mapping and thus give the construction of a decorated hyperbolic structure from an assignment of putative  $\lambda$ -lengths. To this end, consider the topological universal cover  $\tilde{F}$  of  $F$  and the lift  $\tilde{\tau}$  of  $\tau$  to  $\tilde{F}$ ; to each component arc of  $\tilde{\tau}$  is associated the lambda length of its projection.

The proof proceeds by induction, and for the basis step, choose any triangle  $\Delta'_0$  of  $\tilde{\tau}$  and any ideal triangle  $\Delta_0$  in  $\mathbf{H}$ . The ideal vertices of  $\Delta_0$  determine three rays in  $L^+$ , so by Lemma 1, there are three well-defined points in  $L^+$  realizing the putative  $\lambda$ -lengths on the edges of  $\Delta'_0$ . (In effect, this basis step of normalizing a triangle “kills” the conjugacy by the Möbius group in the definition of Teichmüller space.) Of course, the triple of points in  $L^+$  corresponds by affine duality to a triple of horocycles, one centered at each ideal vertex of  $\Delta_0$ , i.e., a “decoration” on  $\Delta_0$ .

To begin the induction step, consider a triangle  $\Delta'_1$  adjacent to  $\Delta'_0$  across an arc in  $\tilde{\tau}$ . The two ideal points which  $\Delta'_0$  and  $\Delta'_1$  share have been lifted to  $u, v \in L^+$  in the basis step, and we let  $w \in L^+$  denote the lift of the third ideal point of  $\Delta'_0$  and consider the plane through the origin determined by  $u, v$ . According to Lemma 2, there is a unique lift  $z \in L^+$  of the third ideal point of  $\Delta'_1$  on the side of this plane not containing the lift of  $w$ , where  $z$  realizes the putative  $\lambda$ -lengths. Again,  $u, v, z$  gives rise via affine duality to another decorated triangle  $\Delta_1$  in  $\mathbf{H}$  sharing one edge and two horocycles with  $\Delta_0$ .

One continues in this manner serially applying Lemma 2 to produce a collection of decorated triangles pairwise sharing edges in  $\mathbf{H}$ , where any two triangles have disjoint interiors (because of our choice of the side of the plane in Lemma 2). Thus, the construction gives an injection  $\tilde{F} \rightarrow \mathbf{H}$ , and we next show that in fact this mapping is also a surjection. To this end, note first that the inductive construction has an image which is open in  $\mathbf{H}$  by construction. According to Lemma 3, there is some  $\varepsilon > 0$  so that each horocyclic arc inside of each triangle has length at least  $\varepsilon$ ; indeed, there are only finitely many values for such lengths because the surface is comprised of finitely many triangles. Thus, each application of the inductive step moves a definite amount along each horocycle, and it follows that the construction has an image which is closed as well. It follows from connectivity of  $\mathbf{H}$  that  $\tilde{F} \rightarrow \mathbf{H}$  is surjective, and furthermore, one can see that  $\tilde{\tau}$  is a tessellation of  $\mathbf{H}$ , i.e., a locally finite collection of geodesics decomposing  $\mathbf{H}$  into ideal triangles.

Following Poincaré, the hyperbolic symmetry group of this tessellation is the required

(normalized) Fuchsian group  $G$  giving a point of Teichmüller space, and the construction likewise provides a decoration on the quotient  $\mathbf{H}/G$  as required. *q.e.d.*

**Remark 3** One thinks of the choice of ideal triangulation as a choice of “basis” for the lambda length coordinates. Formulas for the change of basis are given by Ptolemy transformations, and this leads [7],[8] to a faithful representation of the mapping class group of  $F$  as well as its universal generalization to  $\mathcal{T}ess$ . Furthermore as in Remark 2, the 2-form  $\omega = \sum \omega(a, b, c)$  is invariant under this action and descends to the Weil-Petersson form on Riemann’s moduli space, where the sum is over all triangles complementary to any fixed ideal triangulation and the edges of the triangle have lambda lengths  $a, b, c$  in correct cyclic order determined by an orientation of  $F$ . These two ingredients lead to a natural quantization of Teichmüller space, [3] in the Poisson quantization and [4] in the symplectic quantization.

### 3. Coordinates for circle homeomorphisms

Define a *tesselation*  $\tau$  of the Poincaré disk  $\mathbf{D}$  to be a countable locally finite collection of hyperbolic geodesics in  $\mathbf{D}$  each of whose complementary regions is an ideal triangle. A *distinguished oriented edge* or *doe* of  $\tau$  is the specification of an orientation on one of the geodesics in  $\tau$ . Each geodesic in  $\tau$  has a pair of asymptotes in  $S^1$ , and we let  $\tau^0 \subseteq S^1$  denote the collection of all such asymptotes of geodesics in  $\tau$  and  $\tau^2$  denote the collection of all triangles complementary to  $\cup \tau$ .

Tesselations with doe are “combinatorially rigid” in the following sense. Suppose that  $\tau_1, \tau_2$  are each a tesselation with doe, say the initial and terminal points of the doe in  $\tau_i$  are  $x_i \in \tau_i^0$  and  $y_i \in \tau_i^0$ , respectively, for  $i = 1, 2$ . There is a unique bijection  $f : \tau_1^0 \rightarrow \tau_2^0$  so that  $f(x_1) = x_2, f(y_1) = y_2$ , and whenever  $x, y, z$  in correct cyclic order span an oriented triangle in  $\tau_1^2$ , then  $f(x), f(y), f(z)$  in correct cyclic order also span an oriented triangle in  $\tau_2^2$ . This mapping  $f : \tau_1^0 \rightarrow \tau_2^0$  is called the *characteristic mapping* of the pair of tesselations with doe. In particular, we may fix  $\tau_1 = \tau_*$  to be the Farey tesselation with doe defined in §1, so  $\tau_*^0 = \mathbf{Q}$ . We may thus define the characteristic mapping  $f_\tau : \mathbf{Q} \rightarrow \tau^0$  of the tesselation  $\tau = \tau_2$  with doe.

Define the set

$$\mathcal{T}ess' = \{\text{tesselations with doe of } \mathbf{D}\}.$$

To define a topology on  $\mathcal{T}ess'$ , if  $\tau$  is a tesselation with doe, then we may extend the range of the characteristic mapping  $f_\tau : \mathbf{Q} \rightarrow \tau^0 \subseteq S^1$  to  $S^1$ . The assignment  $\tau \mapsto f_\tau$  determines an embedding of  $\mathcal{T}ess'$  into the function space  $(S^1)^{\mathbf{Q}}$  with the compact-open topology (where  $\mathbf{Q}$  is given its discrete topology), and we endow  $\mathcal{T}ess'$  with the subspace topology.

Define the topological group  $Homeo_+ = Homeo_+(S^1)$  to be the group of all orientation-preserving homeomorphisms of the circle taken with the compact-open topology. If  $f \in Homeo_+$  and  $e$  is any geodesic in  $\mathbf{D}$ , say with ideal points  $x, y \in S^1$ , then define  $f(e)$  to

be the geodesic in  $\mathbf{D}$  with ideal points  $f(x), f(y) \in S^1$ . It is not difficult to see that if  $\tau$  is a tessellation and  $f \in \text{Homeo}_+$ , then  $f(\tau) = \{f(e) : e \in \tau\}$  is also a tessellation. Since a doe on  $\tau$  determines a doe on  $f(\tau)$  in the natural way, there is thus an action of  $\text{Homeo}_+$  on  $\mathcal{Tess}'$ .

**Theorem 5** [8;Theorem 2.3] *The mapping*

$$\begin{aligned} \text{Homeo}_+ &\rightarrow \mathcal{Tess}' \\ f &\mapsto f(\tau_*) \end{aligned}$$

*is a homeomorphism onto.*

**Proof** Injectivity follows from the fact that a homeomorphism is uniquely determined by its values on a dense set. For surjectivity, consider any tessellation with doe  $\tau$ . Using the fact that  $\mathbf{Q}$  and  $\tau^0$  are dense in  $S^1$  and the characteristic mapping  $f_\tau$  is order-preserving by construction, a standard point-set topology argument show that there is a unique orientation-preserving homeomorphism  $f_\tau : S^1 \rightarrow S^1$  which restricts to the characteristic mapping. Both mappings  $f \mapsto f(\tau)$  and  $\tau \mapsto f_\tau$  are continuous by construction. *q.e.d.*

There is the natural diagonal left action of the group  $M\ddot{o}b$  on  $(S^1)^{\mathbf{Q}}$ , which induces a left action of  $M\ddot{o}b$  on the subspace  $\mathcal{Tess}'$ , and we finally define the *universal Teichmüller space*

$$\mathcal{Tess} = M\ddot{o}b \backslash \mathcal{Tess}' \approx M\ddot{o}b \backslash \text{Homeo}_+$$

to be the orbit space with the quotient topology.

A *decoration* on a tessellation  $\tau$  is the specification of horocycles in  $\mathbf{D}$ , one horocycle centered at each point of  $\tau^0$ . Via the affine duality discussed in §1, the characteristic mapping  $f_\tau : \mathbf{Q} \rightarrow S^1$  on a decorated tessellation  $\tau$  with doe extends to a mapping  $g_\tau : \mathbf{Q} \rightarrow L^+$ . The image  $g_\tau(\mathbf{Q})$  is automatically “radially dense” in  $L^+$  in the sense that  $\Pi(g_\tau(\mathbf{Q}))$  is a dense subset of  $S^1$ , where  $\Pi : L^+ \rightarrow S^1$  is the natural projection. Define

$$\widetilde{\mathcal{Tess}}' = \{\text{decorated tessellations } \tau \text{ with doe : } g_\tau(\mathbf{Q}) \text{ is discrete in } L^+\}.$$

The Hausdorff topology on the set of all closed subsets of  $L^+$  induces a subspace topology on the set of all discrete subsets of  $L^+$ , and this in turn induces a compact-open topology on  $\widetilde{\mathcal{Tess}}'$ . There is again a diagonal left action of  $M\ddot{o}b$  by Minkowski isometries on  $\widetilde{\mathcal{Tess}}'$ , and the *decorated universal Teichmüller space* is finally defined to be the topological quotient

$$\widetilde{\mathcal{Tess}} = M\ddot{o}b \backslash \widetilde{\mathcal{Tess}}'.$$

There is the natural forgetful map  $\widetilde{\mathcal{Tess}} \rightarrow \mathcal{Tess}$ , which is evidently continuous.

Given a decorated tessellation  $\tilde{\tau}$  with doe and  $e \in \tau_*$ , there is the corresponding lambda length of the decorated geodesic in  $\tilde{\tau}$  with underlying geodesic  $f_\tau(e)$ , where  $f_\tau$  is the



characteristic mapping of  $\tau$ . Thus, lambda lengths naturally determine an element of  $\mathbf{R}_{>0}^{\tau_*}$ .

**Theorem 6** [8:Theorem 3.1] *The assignment of lambda lengths determines an embedding*

$$\widetilde{\mathcal{T}ess} \rightarrow \mathbf{R}_{>0}^{\tau_*}$$

*onto an open set, where  $\mathbf{R}_{>0}^{\tau_*}$  is given the weak topology (compact-open on  $\mathbf{R}_{>0}^{\tau_*}$  with  $\tau_*$  discrete). Thus,  $\widetilde{\mathcal{T}ess}$  inherits the structure of a Fréchet manifold.*

**Proof** We say that an element  $\tau \in \mathcal{T}ess'$  is *normalized* provided that  $\{\pm 1\} \subseteq \tau^0$ , the doe of  $\tau$  runs from  $-1$  to  $+1$ , and the triangle in  $\tau^2$  lying to the right of the doe coincides with the triangle spanned by  $-1, +1, -\sqrt{-1} \in S^1$ . Since  $Möb$  acts three-effectively on  $S^1$  and the value of a Möbius transformation at three points of  $S^1$  determines it uniquely, each  $Möb$ -orbit on  $\mathcal{T}ess'$  admits a unique normalized representative.  $\mathcal{T}ess$  is thus canonically identified with the collection of all normalized tessellations. (Again, we have “killed” the Möbius group by normalization.)

To define a left inverse to the assignment  $\lambda \in \mathbf{R}_{>0}^{\tau_*}$  of lambda lengths, use Lemma 1 to uniquely lift the vertices  $\pm 1, -\sqrt{-1}$  of the triangle of  $\tau_*$  to the right of the doe to points in the rays in  $L^+$  lying over these vertices realizing the lambda lengths. As in the proof of Theorem 4, we may then uniquely extend using Lemma 2 to a function  $g : \mathbf{Q} \rightarrow L^+$  realizing the lambda lengths.

If  $g(\mathbf{Q}) \subseteq L^+$  is radially dense, then the order-preserving mapping  $\mathbf{Q} \rightarrow L^+ \rightarrow S^1$  interpolates a unique homeomorphism  $f : S^1 \rightarrow S^1$  as before. One can always produce a discrete decoration, say by taking the point  $f(p) \in L^+$  to have height  $i$  in  $\mathbf{R}^3$  if  $p \in \mathbf{Q}$  is of Farey generation  $i$ .

It follows that the mapping  $\widetilde{\mathcal{T}ess} \rightarrow \mathbf{R}_{>0}^{\tau_*}$  is indeed injective. Continuity follows from the definition of the topologies, and openness of the image follows from the construction. *q.e.d.*

Recall [1] that a *quasi-symmetric* homeomorphism of  $S^1$  is the restriction to  $S^1$  of a quasi-conformal homeomorphism of  $\mathbf{D}$ , and let  $Homeo_{qs} \subseteq Homeo_+$  denote the subspace of all quasi-symmetric homeomorphisms of the circle. *Bers' universal Teichmüller space* [2] is the quotient

$$Möb \backslash Homeo_{qs} \subseteq Möb \backslash Homeo_+ \approx \mathcal{T}ess$$

and is highly studied.

As is usual in these circumstances, it is difficult to explicitly characterize the image  $\widetilde{\mathcal{T}ess} \subseteq \mathbf{R}_{>0}^{\tau_*}$ . On the other hand, there are the following useful characterizations of subsets of  $\widetilde{\mathcal{T}ess} \subseteq \mathbf{R}_{>0}^{\tau_*}$ .

We say that  $\lambda \in \mathbf{R}_+^{\tau_*}$  is *pinched* provided there is some real number  $K > 1$  so that

$$K^{-1} \leq \lambda(e) \leq K, \text{ for each } e \in \tau_*.$$

**Theorem 7** [8;Theorem 6.3] *If  $\lambda \in \mathbf{R}_{>0}^{\tau*}$  is pinched, then there is a decorated tessellation whose lambda lengths are given by  $\lambda$ , and the corresponding subset of  $L^+$  is discrete and radially dense.*

**Theorem 8** [8;Theorem 6.4 (joint with Sullivan)] *If  $\lambda \in \mathbf{R}_{>0}^{\tau*}$  is pinched, then the corresponding homeomorphism of the circle is quasi-symmetric.*

In particular, consider any decoration on a marked punctured Riemann surface  $F$  uniformized by  $G < \text{Möb}$ . Choose an ideal triangulation of  $\mathbf{D}/G$ , and lift it to a tessellation  $\tau$  of  $\mathbf{D}$  which inherits a  $G$ -invariant decoration. Choose a doe of  $\tau$  to determine a point of  $\widetilde{\mathcal{T}ess}$ . The lambda lengths in  $F$  lift to  $G$ -invariant lambda lengths on  $\tau$ , and they are pinched since they take only finitely many values. Furthermore, if  $G < PSL_2$  is finite-index and free of elliptics and  $\phi : \mathbf{D} \rightarrow \mathbf{D}$  is any  $G$ -invariant quasi-conformal map conjugating  $G$  to an isomorphic group, then we claim that the boundary values of  $\phi$  satisfy the smoothness conditions above. To see this, conjugate in domain and range so that corresponding parabolic covering transformations are each given in upper halfspace by  $z \mapsto z + 1$  (thus, destroying the normalization in  $\widetilde{\mathcal{T}ess}$ ), so the monotone function  $\phi(t)$  nearly agrees with the integral part of  $t$ . It follows directly that  $\phi(t)$  is differentiable at each point of  $\mathbf{Q}$ , and the derivatives at points of  $\mathbf{Q}$  are uniformly near unity. (Compare with [13].)

In contrast, a quasi-symmetric map  $\phi : S^1 \rightarrow S^1$  arising from pinched lambda lengths need not have these differentiability properties at  $\mathbf{Q}$ . To see this, use the formula in the Introduction for lambda lengths in the upper halfspace model to produce pinched lambda lengths so that the two one-sided derivatives at infinity disagree or even fail to exist.

This corrects the second part of Theorem 6.4 from [8]. For the corresponding result in the setting of the solenoid, see Proposition 12 in §4.

#### 4. Coordinates for the solenoid

Let  $G < PSL_2$  be any finite-index subgroup, and choose a base-point in the quotient surface or orbifold  $M = \mathbf{D}/G$ ; in particular, for  $G = PSL_2$ ,  $M$  is the orbifold modular curve. Consider the category  $\mathcal{C}_M$  of all finite-sheeted orbifold covers  $\pi : F \rightarrow M$ , where  $F$  is a punctured Riemann surface.  $\mathcal{C}_M$  is a directed set, where  $\pi_1 \leq \pi_2$  if there is a finite-sheeted unbranched cover  $\pi_{2,1} : F_2 \rightarrow F_1$  of Riemann surfaces so that the following diagram commutes:

$$\begin{array}{ccc} F_2 & \xrightarrow{\pi_{2,1}} & F_1 \\ \pi_2 \searrow & & \swarrow \pi_1 \\ & M & \end{array}$$

In other words by covering space theory, if  $\Gamma_i < G < PSL_2$  uniformizes  $F_i$  for  $i = 1, 2$ ,

then  $\pi_1 \leq \pi_2$  if and only if  $\Gamma_1$  is a finite-index subgroup of  $\Gamma_2$ .

A topological space, the *punctured solenoid*, is defined in analogy to [12] to be the inverse limit

$$\mathcal{H}_M = \varprojlim \mathcal{C}_M;$$

a point of  $\mathcal{H}_M$  is thus determined by choices of points  $y_i \in F_i$  for each cover  $\pi_i : F_i \rightarrow M$ , where the choices are “compatible” in the sense that if  $\pi_1 \leq \pi_2$ , then we have in the notation above  $\pi_{2,1}(y_2) = y_1$ .

Since punctured surface groups are cofinal in the set of punctured orbifold groups, we could have equivalently considered the category of orbifold covers of  $M$  in the definition of  $\mathcal{H}_M$ . Furthermore, if  $\Gamma < G$  is of finite-index, then  $\mathcal{H}_\Gamma$  is naturally homeomorphic to  $\mathcal{H}_G$ , and we may thus think of *the* punctured solenoid  $\mathcal{H} = \mathcal{H}_{PSL_2}$

One can from first principals develop the Teichmüller theory of  $\mathcal{H}$  along classical lines [12], [5], [11]. Instead, we next introduce an explicit space homeomorphic to  $\mathcal{H}$  following [5], and we shall then *define* the Teichmüller space representation theoretically in analogy to punctured surfaces in §1.

$G$  has characteristic subgroups

$$G_N = \cap \{\Gamma < G : [\Gamma : G] \leq N\},$$

for each  $N \geq 1$ , and these are nested  $G_N < G_{N+1}$ . Define a metric

$$\begin{aligned} G \times G &\rightarrow \mathbf{R}, \\ \gamma \times \delta &\mapsto \min\left\{\frac{1}{N} : \gamma\delta^{-1} \in G_N\right\}, \end{aligned}$$

and define the *pro-finite completion*  $\hat{G}$  of  $G$  as a space to be the metric completion of  $G$ , i.e., suitable equivalence classes of Cauchy sequences in  $G$ . Termwise multiplication of Cauchy sequences gives  $\hat{G}$  the structure of a topological group, and termwise multiplication by  $G$  gives a continuous action of  $G$  on  $\hat{G}$ .

For any sub-group  $G < PSL_2$  of finite-index, we may define the quotient

$$\mathcal{H}_G = \mathbf{D} \times_G \hat{G} = (\mathbf{D} \times \hat{G})/G,$$

where  $\gamma \in G$  acts by

$$\begin{aligned} \gamma : \mathbf{D} \times \hat{G} &\rightarrow \mathbf{D} \times \hat{G} \\ (z, t) &\mapsto (\gamma z, t\gamma^{-1}). \end{aligned}$$

**Lemma 9** [5]  *$\mathcal{H}$  is homeomorphic to  $\mathcal{H}_G$  for any  $G < PSL_2$  of finite-index.*

In particular, each path component, or “leaf”, of  $\mathcal{H}$  is homeomorphic to a disk (by residual finiteness of  $G$ ), and each leaf is dense in  $\mathcal{H}$  (since  $G$  is dense in  $\hat{G}$ ).

Let us for definiteness simply fix  $G = PSL_2$  and consider the collection  $Hom'(G \times \hat{G}, Möb)$  of all functions  $\rho : G \times \hat{G} \rightarrow Möb$  satisfying the following three properties:

Property 1:  $\rho$  is continuous;

Property 2: for each  $\gamma_1, \gamma_2 \in G$  and  $t \in \hat{G}$ , we have

$$\rho(\gamma_1 \circ \gamma_2, t) = \rho(\gamma_1, t\gamma_2^{-1}) \circ \rho(\gamma_2, t);$$

Property 3: for every  $t \in \hat{G}$ , there is a quasi-conformal mapping  $\phi_t : \mathbf{D} \rightarrow \mathbf{D}$  so that for every  $\gamma \in G$ , the following diagram commutes:

$$\begin{array}{ccc} (z, t) & \mapsto & (\gamma z, t\gamma^{-1}) \\ \mathbf{D} \times \hat{G} & \rightarrow & \mathbf{D} \times \hat{G} \\ \phi_t \times \text{id} \downarrow & & \downarrow \phi_{t\gamma^{-1}} \times \text{id} \\ \mathbf{D} \times \hat{G} & \rightarrow & \mathbf{D} \times \hat{G} \\ (z, t) & \mapsto & (\rho(\gamma, t)z, t\gamma^{-1}) \end{array}$$

Furthermore,  $\phi_t$  varies continuously in  $t \in \hat{G}$  for the common refinement of the  $C^\infty$  topology of uniform convergence on compacta in  $\mathbf{D}$  and the usual topology on Bers' universal Teichmüller space  $Möb \backslash Homeo_{qs}$  of the extension of  $\phi_t$  to the circle at infinity.

As to Property 1, notice that since  $G$  is discrete,  $\rho$  is continuous if and only if it is so in its second variable only. Property 2 is a kind of homomorphism property of  $\rho$  mixing the leaves; notice in particular that taking  $\gamma_2 = I$  shows that  $\rho(I, t) = I$  for all  $t \in \hat{G}$ . Property 3 mandates that for each  $t \in \hat{G}$ ,  $\phi_t$  conjugates the standard action of  $G$  on  $\mathbf{D} \times \hat{G}$  at the top of the diagram to the action

$$\gamma_\rho : (z, t) \mapsto (\rho(\gamma, t)z, t\gamma^{-1})$$

at the bottom, and we let  $G_\rho = \{\gamma_\rho : \gamma \in G\} \approx G$ . Notice that the action of  $G_\rho$  on  $\mathbf{D} \times \hat{G}$  extends continuously to an action on  $S^1 \times \hat{G}$ . We finally define the solenoid (with marked hyperbolic structure)

$$\mathcal{H}_\rho = (\mathbf{D} \times_\rho \hat{G}) = (\mathbf{D} \times \hat{G})/G_\rho.$$

The collection  $\phi_t$ , for  $t \in \hat{G}$ , thus induces a homeomorphism  $\mathcal{H} \rightarrow \mathcal{H}_\rho$ .

Define the group  $Cont(\hat{G}, Möb)$  to be the collection of all continuous maps  $\alpha : \hat{G} \rightarrow Möb$ , where the product of two  $\alpha, \beta \in Cont(\hat{G}, Möb)$  is taken pointwise  $(\alpha\beta)(t) = \alpha(t) \circ \beta(t)$  in  $Möb$ .  $\alpha \in Cont(\hat{G}, Möb)$  acts continuously on  $\rho \in Hom'(G \times \hat{G}, Möb)$  according to

$$(\alpha\rho)(\gamma, t) = \alpha^{-1}(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha(t).$$

**Theorem 10** [10] *There is a natural homeomorphism of the Teichmüller space of the solenoid  $\mathcal{H}$  with*

$$T(\mathcal{H}) = Hom'(G \times \hat{G}, Möb) / Cont(\hat{G}, Möb).$$

Rather than describe the proof here, we shall for simplicity simply take this quotient as the definition of the Teichmüller space  $T(\mathcal{H})$ . Again with an eye towards simplicity here, rather than define punctures of solenoids intrinsically (as suitable equivalence classes of ends of escaping rays), we can more simply proceed as follows. Each  $\phi_t : \mathbf{D} \rightarrow \mathbf{D}$  extends continuously to a quasi-symmetric mapping  $\phi_t : S^1 \rightarrow S^1$ . We say that a point  $(p, t) \in S^1 \times \hat{G}$  is a  $\rho$ -puncture if  $\phi_t^{-1}(p) \in \mathbf{Q}$ , and a puncture of  $\mathcal{H}_\rho$  itself is a  $G_\rho$ -orbit of  $\rho$ -punctures. A  $\rho$ -horocycle at a  $\rho$ -puncture  $(p, t)$  is the image under  $\phi_t$  of a horocycle in  $\mathbf{D}$  centered at  $\phi_t^{-1}(p)$ .

A *decoration* on  $\mathcal{H}_\rho$ , or a “decorated hyperbolic structure” on  $\mathcal{H}$ , is a function

$$\tilde{\rho} : G \times \hat{G} \times \mathbf{Q} \rightarrow Möb \times L^+,$$

where

$$\tilde{\rho}(\gamma, t, q) = \rho(\gamma, t) \times h(t, q)$$

with  $\rho(\gamma, t) \in Hom'(G \times \hat{G}, Möb)$ , which satisfies the following conditions:

Property 4: for each  $t \in \hat{G}$ , the image  $h(t, \mathbf{Q}) \subseteq L^+$  is discrete and radially dense;

Property 5: for each  $q \in \mathbf{Q}$ , the restriction  $h(\cdot, q) : \hat{G} \rightarrow L^+$  is continuous;

Property 6:  $\tilde{\rho}$  is  $G$ -invariant in the sense that if  $\delta \in G$ , then

$$\delta \circ \tilde{\rho}(\gamma, t, q) = \tilde{\rho}(\delta\gamma, t\delta^{-1}, \delta q),$$

where  $\delta$  acts diagonally  $\delta : (\gamma, q) \mapsto (\delta\gamma, \delta q)$  on  $Möb \times L^+$  with  $\delta q$  the natural action of  $G = PSL_2$  on  $L^+$ .

Finally, let  $Hom'(G \times \hat{G} \times \mathbf{Q}, Möb \times L^+)$  denote the space of all decorated hyperbolic structures on  $\mathcal{H}$  satisfying the properties above, and define the *decorated Teichmüller space* as the quotient

$$\tilde{T}(\mathcal{H}) = Hom'(G \times \hat{G} \times \mathbf{Q}, Möb \times L^+) / Cont(\hat{G}, Möb),$$

where  $\alpha : \hat{G} \rightarrow \text{Möb}$  acts on  $\tilde{\rho}$  by

$$(\alpha\tilde{\rho})(\gamma, t, q) = (\alpha^{-1}(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha(t)) \times (h(t, \alpha(t)q)).$$

Forgetting decoration induces a continuous surjection  $\tilde{T}(\mathcal{H}) \rightarrow T(\mathcal{H})$ .

There is a natural mapping  $\lambda : \tilde{T}(\mathcal{H}) \rightarrow (\mathbf{R}_{>0}^{\tau_*})^{\hat{G}}$  which assigns to a function  $\tilde{\rho} : G \times \hat{G} \times \mathbf{Q} \rightarrow \text{Möb} \times L^+$  the lambda length for the  $G_\rho$  metric of the  $\rho$ -horocycles determined by  $h$  at the endpoints of the geodesic in  $\mathcal{H}_\rho$  labeled by  $\gamma$ .

**Theorem 11** [10] *The assignment of lambda lengths determines an embedding*

$$\tilde{T}(\mathcal{H}) \rightarrow \text{Cont}(\hat{G}, \mathbf{R}_{>0}^{\tau_*})$$

onto an open set, where we take the strong topology on  $\mathbf{R}_{>0}^{\tau_*}$  and on  $\text{Cont}(\hat{G}, \mathbf{R}_{>0}^{\tau_*})$ .

**Sketch of Proof** To prove the mapping is injective, we must again define the construction of decorated hyperbolic structure from a continuous  $\lambda_t \in (\mathbf{R}_{>0}^{\tau_*})^{\hat{G}}$ . To this end, begin the definition of  $\tilde{\rho} = \rho \times h$  on the triangle to the right of the doe in  $\tau_*$  with lambda lengths given by  $\lambda_t$ . As usual according to Lemma 1, we can uniquely lift to a triple of points in  $L^+$  lying over  $\pm 1, -\sqrt{-1}$ .

It is easily seen from the identification of  $G = PSL_2$  with the oriented edges of  $\tau_*$  that any  $\gamma \in PSL_2$  can be written uniquely in the one of the following forms:

- i)  $\gamma = I$ ;
- ii)  $\gamma$  lies in the free semi-group generated by either  $U, T$  or  $U^{-1}, T^{-1}$ ;
- iii)  $\gamma$  arises from either i) or ii) by addition of the prefix  $S$ .

To define  $\rho(\gamma, t) \in \text{Möb}$ , we shall specify an ideal triangle-doe pair  $(\Delta', e')$  in  $\mathbf{D}$ , where  $e''$  is oriented with  $\Delta'$  to its right. There is then a unique  $\rho \in \text{Möb}$  mapping to the vertices of  $\Delta'$  the vertices  $\pm 1, -\sqrt{-1}$  of the triangle  $\Delta$  to the right of the doe  $e_I$  in  $\tau_*$  and mapping  $e_I$  to  $e'$ .

Let us write  $\gamma \in G$  in one of the forms i-iii) above. Of course, if  $\gamma = I$ , then  $\rho(\gamma, t) = I$  as follows from the functional equation in Property 2, and we take  $(\Delta', e') = (\Delta, e_I)$ .

If  $\gamma = S$ , then let us begin with the lambda lengths  $\lambda_t \in \mathbf{R}_{>0}^{\tau_*}$  on the edges of  $\Delta$  and employ Lemma 1 to uniquely realize a lift to  $L^+$  of the vertices of this decoration on the triangle  $\Delta$ . On the triangle to the left of the doe, consider the lambda lengths  $\lambda_{tS^{-1}} = \lambda_{tS}$ . It need not be that  $\lambda_t(e_I) = \lambda_{tS}(e_I)$ , and we re-scale, taking lambda lengths

$$\frac{\lambda_t(e_I)}{\lambda_{tS}(e_I)} \lambda_{tS}(\cdot)$$

on the edges  $e_U, e_T$ . This defines lambda lengths on the edges of the quadrilateral in  $\tau_*$  triangulated by the doe. Again using Lemma 2, there is a unique lift of  $\sqrt{-1}$  to  $L^+$  realizing the lambda lengths, and the projection of this point to  $S^1$  is one of the vertices of  $\Delta'$ . The other two vertices of  $\Delta'$  are  $\pm 1$ , and the doe is  $e' = e_S$ , completing the definition in this case. Notice that this element of  $M\ddot{o}b$  that maps  $(\Delta, e_I) \rightarrow (\Delta', e_S)$  is necessarily involutive.

The case of any word in one of the semi-groups in ii) is handled by induction on the length, where for instance for any such  $\gamma$  that has a prefix  $U$ , one begins from the lambda lengths  $\lambda_t$  on  $\Delta$  and uses the re-scaled lambda lengths  $\lambda_{tU^{-1}}$  on the edges  $e_U, e_T$ ; the edge corresponding to  $e_U$  is the doe of the first step.

The remaining case iii) of a word from one of the semi-groups with prefix  $S$  is handled in exactly the same manner completing the definition of the construction of  $\rho : G \times \hat{G} \rightarrow M\ddot{o}b$ . The functional equation in Property 2 on  $\rho$  follows by definition. Furthermore, since  $\lambda_t \in \mathbf{R}_{>0}^{\tau_*}$  depends continuously on  $t$  (because of the definition of the topology),  $\rho(\gamma, t)$  is also continuous in  $t$  as required in Property 1; indeed, the entries of  $\rho(\gamma, t) \in M\ddot{o}b$  are algebraic function of finitely many lambda lengths.

As for Property 3 in the definition of  $T(\mathcal{H})$ , we claim that for each  $t \in \hat{G}$ ,  $\lambda_t \in \mathbf{R}_{>0}^{\tau_*}$  is pinched if  $\lambda \in Cont(\hat{G}, \mathbf{R}_{>0}^{\tau_*})$ .

To see this, note that the very definition of a continuous function  $\lambda : \hat{G} \rightarrow \mathbf{R}_{>0}^{\tau_*}$  means that  $\forall K \exists N \forall e \in \tau_* \forall \gamma \in G_N$ , we have

$$1 + K^{-1} \leq \frac{\lambda_t(e)}{\lambda_t(\gamma e)} \leq 1 + K.$$

Take say  $K = \frac{1}{2}$  and its corresponding  $N$ . A fundamental domain for  $G_N$  has only a finite collection of values of lambda lengths, and any other lambda length is at most a factor of  $3/2$  times a lambda length this finite set, and at least a factor of  $1/2$  times a lambda length in this finite set.  $\lambda_t$  is therefore pinched, proving the claim.

It follows that for each  $t \in \hat{G}$ ,  $\lambda_t : \tau_* \rightarrow \mathbf{R}_{>0}$  is necessarily pinched. By Theorem 8, there is a corresponding quasi-conformal homeomorphism  $\phi_t : \mathbf{D} \rightarrow \mathbf{D}$ . Commutativity of the diagram and continuity in Property 3 follow by construction, and this completes the proof that the function  $\rho$  constructed above lies in  $Hom'(G \times \hat{G}, M\ddot{o}b)$  and hence determines a point of  $T(\mathcal{H})$ .

To define a decoration on the  $\rho$ -punctures, each  $\lambda_t$  determines a decoration on  $\tau_* \times \{t\} \subseteq \mathbf{D} \times \hat{G}$ , as required. Property 4 is guaranteed by the claim and Theorem 7. Property 5 holds as before in the strong sense that the Euclidean coordinates of each  $h(t, q)$  are algebraic functions of finitely many lambda lengths, and Property 6 holds by invariance of lambda lengths under  $M\ddot{o}b$ . *q.e.d.*

**Proposition 12** [10] *Suppose that  $\lambda \in Cont(\hat{G}, \mathbf{R}_{>0}^{\tau_*})$ . Then for each  $t \in \hat{G}$ ,  $\lambda_t : \tau_* \rightarrow \mathbf{R}_{>0}$  corresponds to a quasi-conformal homeomorphism  $\phi_t : \mathbf{D} \rightarrow \mathbf{D}$  whose quasi-symmetric*

extension  $\phi_t : S^1 \rightarrow S^1$  is differentiable at each point of  $\mathbf{Q}$  with derivative uniformly near unity.

**Proof** As above, continuity of  $\lambda_t$  in  $t$  implies that each  $\lambda_t$  is pinched, which gives a quasi-symmetric map  $\phi_t$ , for  $t \in \hat{G}$ , by Theorem 8. Using the upper halfspace model, normalize  $\phi_t$  such that it fixes 0 and  $\infty$ , whence the geodesics of  $\tau_*$  that limit to  $\infty$  are mapped by  $\phi_t$  onto geodesics which likewise limit to  $\infty$ . Again by continuity, we conclude that for each  $\epsilon > 0$ ,  $e \in \tau_*$  and  $\gamma \in PSL_2$  fixing  $\infty$ , we have  $|\lambda_t(e) - \lambda_t(\gamma^n e)| < \epsilon$  for  $n$  sufficiently large. The differences  $a_n = \phi_t(n) - \phi_t(n-1)$  are then  $\epsilon_1$  close using continuity of the assignment in Lemmas 1 and 2 of decorated ideal triangles given lambda lengths.

We show that  $\lim_{n \rightarrow \infty} \frac{1}{n} \phi_t(n)$  exists and is bounded, which proves the proposition. To this end since  $|a_i - a_{nk+i}| < \epsilon_1$  for all  $i, k$ , we find

$$\left| \frac{1}{n}(a_1 + \cdots + a_n) - \frac{1}{nk}(a_1 + \cdots + a_{nk}) \right| \leq 1/n \sum_{i=1}^n \left| a_i - \frac{1}{k}(a_i + a_{i+n} + \cdots + a_{i+n(k-1)}) \right| \leq \epsilon_1,$$

and it follows that  $\frac{1}{n} \phi_t(n)$  is a Cauchy sequence, as desired. *q.e.d.*

Differentiability at the rational points, which holds in the case of punctured surfaces (cf. the discussion following Theorem 8) thus also holds for the solenoid according to Proposition 12. Differentiability does not hold for general decorated tessellations with pinched lambda lengths however (cf. the discussion and counter-example following Theorem 8).

## 5. Concluding remarks

In addition to the function  $h : \hat{G} \times \mathbf{Q} \rightarrow L^+$  constructed in the proof of Theorem 11, there is another natural function  $h_1 : \hat{G} \times \mathbf{Q} \rightarrow L^+$  defined as follows. Begin with the “standard” decoration on  $\tau_*$  where all lambda lengths are  $\sqrt{2}$ . By Proposition 12, each  $\phi_t$  is smooth with bounded derivative  $\phi'_t(q)$  at each point of  $\mathbf{Q} \subseteq S^1$ . In an upper halfspace model of  $\mathbf{H}$  with the endpoint of the doe at infinity, scale the Euclidean diameter of the horocycle centered at  $q \in \mathbf{Q}$  by the absolute value of the derivative  $|\phi'_t(q)|$  to determine the diameter of the horocycle at  $\phi_t(q)$ . This defines the function  $h_1(t, q)$ . Notice that  $h_1(t, \mathbf{Q})$  is again discrete and radially dense in  $L^+$ , but there seems to be no guarantee that the function  $h_1$  discussed above satisfies Property 5.

The representation-theoretic treatment of the Teichmüller theory of the punctured solenoid seems to us quite appealing. For example, the strong topology for the solenoid (in Theorem 11) in contrast to the weak topology for circle homeomorphisms (in Theorem 6) is interesting. Furthermore, one can naturally induce stronger or weaker transverse structures in the  $\hat{G}$  direction in  $\mathcal{H}$  by imposing conditions other than continuity on the lambda length functions in Theorem 11, and we wonder in particular what is the transverse regularity of the function  $h_1$ .



As was mentioned in the Introduction, Theorem 11 is the first step of an ongoing program [10] to develop the decorated Teichmüller theory of the punctured solenoid. Though there is an alternative to the construction of  $\rho : G \times \hat{G} \rightarrow \text{Möb}$  in the proof of Theorem 11, the argument given here bears a close relation to the treatment of broken hyperbolic structures in [6].

Lambda lengths enjoy the simple transformation property of Ptolemy transformations (cf. Remark 1), and furthermore, there is a simple invariant two-form (cf. Remark 2). These ingredients have been used [3],[4] to give a quantization of classical Teichmüller theory (cf. Remark 3). These same two ingredients persist for the universal Teichmüller theory as well as for the punctured solenoid and might be used to quantize these Teichmüller theories as well.

It is an open (but not centrally important) problem in number theory to calculate the index of  $G_N$  in  $G = PSL_2$ , and an algorithm for its calculation devolves to the “cubic fatgraph enumeration problem” for surfaces of fixed Euler characteristic arising as total spaces of covers of  $\mathbf{D}/G$  of degree  $N$ . More speculatively, there is a natural group [10] generated by the  $G_N$ -equivariant Ptolemy moves for some  $N$ , which is closely related to the completions [9] of the universal Ptolemy group studied in the context of Grothendieck absolute Galois theory.

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